BOHR International Journal of Operations Management Research and Practices
2024, Vol. 3, No. 1, pp. 8–14
DOI: 10.54646/bijomrp.2024.23
www.bohrpub.com

RESEARCH

Applying fuzzy theory to develop linguistic control charts–The pLCC model

Phong Nguyen Nhu[1]* and Tu Anh Nguyen Nhu[2]

[1]Industrial Systems Engineering, University of Technology, VNU-HCM, Melbourne Australia
[2]Information Technology, Monash University, Melbourne, Australia

***Correspondence:**
Phong Nguyen Nhu,
nnphong@hcmut.edu.vn

Received: 09 November 2023; **Accepted:** 16 December 2023; **Published:** 12 January 2024

This paper studies an approach to use fuzzy set theory and possibility theory to construct control charts – a very important on-line process control tool used in quality control. The control chart is constructed based on linguistic data. The model aims to control the process simply and effectively. The quality characteristics are modeled by fuzzy variable. The status of quality characteristics is modeled by triangle fuzzy number. Fuzzy arithmetic is used to calculate the control chart's center line. The control chart's control limits are constructed according to Shewhart's principle. The fuzzy variable's standard deviation is calculated by a model developed by Kauffman and Gupta.

Keywords: quality control, Shewhart's control chart, fuzzy set theory, possibility theory, fuzzy arithmetic, triangle fuzzy number, linguistic control chart, fuzzy variable

1. Introduction

Quality control ensures that product quality characteristics are at nominal or desired levels, including process control and acceptance sampling. Control charts are important tools of on-line process control in Quality Control, including Variable Control Charts VCCs and Attribute Control Charts ACCs.

VCCs controls variable quality characteristics in the form of numerical measurements. ACC controls attribute quality characteristics that cannot be expressed as an arithmetic quantity. In general, ACCs are simple but not as sensitive as VCCs in detecting process shifts.

ACCs are based on attribute data observed from the process; products are classified only into 2 states, good to accept and bad to reject. This classification may not be rational to show exactly the product quality, which may change continuously from bad to reject to good to accept. In this case, the quality level of products could be evaluated by many linguistic values like bad, poor, medium, good, and excellent. The quality characteristics are now modeled by a linguistic variable; thus, the control chart is called linguistic control chart LCC.

LCCs are more effective than traditional control charts in terms of quality costs (i.e., failure costs, appraisal costs, prevention costs). LCCs reduce quality costs via reducing failure costs and appraisal costs. LCCs are simpler than VCCs: instead of using special facilities, LCCs access the quality level of products by experience of expert, resulting in reducing appraisal costs. On the other hand, LCCs are more sensitive than ACCs since they access product quality by more than 2 levels as in ACC cases, resulting in reducing failure costs.

This article develops a model of linguistic control chart, pLCC, based on the concepts of fuzzy variables, called linguistic variables, and Shewhart's control charts. The computing method is based on Fuzzy Arithmetic with the objective of making calculation simple.

2. Literature review

2.1. Shewhart's control charts

Control charts are run charts showing the relationship between quality characteristics and time described by samples (1), consisting of center line CL and control limits. Center line CL is the average value of the quality characteristics while the process is in control. Control limits include Upper Control Limit UCL and Lower Control Limit LCL.

When the samples are inside the control limits, the process is considered in control. When the samples are outside the control limits, the process is considered out of control and needs to be investigated, with actions taken to eliminate the assignable causes in order to bring the process back in control.

Let V be the statistics for the quality characteristics under consideration. Let the expected value and standard deviation of V be μ_V and σ_V, respectively (2). According to Shewhart's principle, the center line and control limits of the control charts are as follows:

$$UCL = \mu_V + L\sigma_V$$

$$CL = \mu_V$$

$$LCL = \mu_V - L\sigma_V$$

where L is the factor showing the relative distance between CL and UCL, LCL. The *distance factor L* is often defined by the *probability of type 1 error* α while knowing the distribution of the quality characteristics.

2.2. Applying fuzzy theories in developing control charts

There are many studies in using fuzzy theories in quality control (3). Williams and Zigli (1987) argued strongly for quality control techniques that recognize and incorporate the imprecision of human judgment. The vagueness and ambiguity inherent in linguistic variables may be treated mathematically with the help of fuzzy set theory introduced by Zadeh (1965). According Bradsaw (1983), constructing control limits based on fuzzy set theory is more realistic in process control. Kawowski & Evans (1986) proposed an approach of using linguistic variables in modeling quality characteristics and using fuzzy numbers in constructing control limits.

Wang & Raz constructed linguistic control charts in 1989 (4). Kanawaga, Tamaki & Ohta proposed new LCCs based on probability distribution in 1993 (5). Fiorenzo Franceschini and Daniele Romano developed a model of linguistic control chart, based on linguistic quantifiers in 1999 (6). Murat Gulbay et al. constructed an α-cut linguistic control chart in 2004 (7).

The Wang & Raz models do not show specific computing method to construct control charts. In addition, the models do not use the distribution of the quality characteristics to do the sensitivity analysis to help construct control charts. The Kanawaga model has solved the weakness of the Wang & Raz models, but it is very complicated and does not analyze the process shift in case of attribute quality characteristics. This article proposes a simple model for constructing linguistic control charts.

2.3. Triangular fuzzy numbers

Fuzzy numbers are fuzzy sets defined on the set of real numbers. Didier Dubois and Henry Prade formulated flat fuzzy numbers (8). From flat fuzzy numbers, P.J. Macvicar-Whelan builds a trapezoidal fuzzy number with 4 parameters. Triangular fuzzy numbers are a special type of trapezoidal fuzzy numbers.

A triangular fuzzy number A(a, b, c), as shown in **Figure 1**, has the membership function of the following form:

$$\mu_A(x) = \begin{cases} 0, & x < a-b \\ (x-a+b)/b, & a-b \le x \le a \\ (a+c-x)/c, & a < x \le a+c \\ 0, & a+c \; x \end{cases}$$

According to triangular fuzzy numbers' properties, if A and B are two triangular fuzzy numbers, then A+B is also a triangular fuzzy number:

$$\begin{cases} A = (a_1, a_2, a_3) \\ B = (b_1, b_2, b_3) \end{cases}$$

$$\Rightarrow A + B = (a_1 + b_1, a_2 + b_2, a_3 + c_3)$$

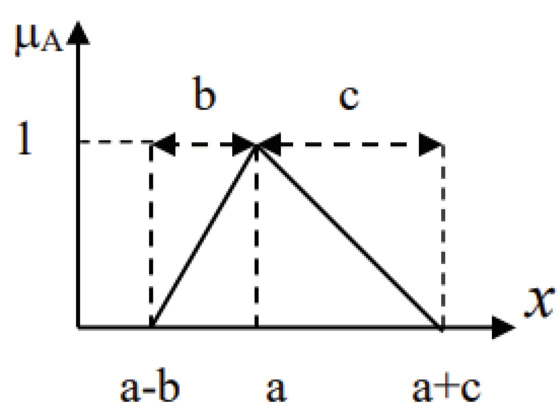

FIGURE 1 | Triangular fuzzy number.

If A is a triangular fuzzy number, and c is a positive real number, then cA is also a triangular fuzzy number:

$$\begin{cases} A = (a_1, a_2, a_3) \\ c \in R^+ \end{cases}$$

$$\Rightarrow cA = (ca_1, ca_2, ca_3)$$

2.4. Fuzzy variables

In possibility theory (8), fuzzy variables take values of fuzzy numbers and distributions of fuzzy variables are possibilistic distributions. The possibilistic distribution π of a fuzzy variable is the membership function μ of the corresponding fuzzy number.

Let V be a fuzzy variable in the set X with possibilistic distribution π. The expected value of V could be defined as follows:

$$\mu_V = \{x|\pi(x) = 1\}, \ \forall x \in X$$

The standard deviation of V could be defined by the formula developed by Kaufman & Gupta (1985)*:*

$$\sigma_V = \int_0^1 [\pi_r(\alpha) - \pi_l(\alpha)]\, d\alpha$$

$$\pi_r(\alpha) = Sup[\pi(\alpha)]$$

$$\pi_l(\alpha) = \mathrm{Inf}[\pi(\alpha)]$$

If V is a triangular variable V (a, b, c), then the expected value and the standard deviation of V are as follows.

$$\mu_V = a$$

$$\sigma_V = (b + c)/2$$

2.5. Linguistic variables

Linguistic variables are variables that take linguistic values in linguistic set *T*.

$$T = \{L_i, \ i = 1 \div t\}$$

where t is the number of linguistic values and L_i are linguistic values. Linguistic values are often defined by linguistic quality levels like excellence, good, bad, . . .

Linguistic variables are fuzzy variables; therefore, linguistic values could be modeled by fuzzy numbers in a based set X that is the set of quality level of the quality characteristics under control.

3. Research methodology

The research methodology for constructing the Linguistic Control Chart pLCC models is shown by the procedure, including the following steps:

- Step 1: Define quality levels of the linguistic variables
- Step 2: Collect data on the quality characteristics under control
- Step 3: Construct the fuzzy variable of sample mean $\overline{X}_j$
- Step 4: Identify the fuzzy variable of grand sample means
- Step 5: Identify the center line CL
- Step 6: Identify the standard deviation of the grand sample mean
- Step 7: Construct the control limits UCL and LCL
- Step 8: Identify the sample points $\overline{X}_j$ in the chart
- Step 9: Assess the control status of the process

3.1. Step 1: Define quality levels of the linguistic variables

The pLCC model standardizes the base set X as a set of unit range, meaning the larger the value of X, the higher the quality level, and 0 means the worst quality level and 1 means the best quality level.

$$X = [0, 1]$$

With the goal of simple calculation, the linguistic set *T* is defined by 5 linguistic levels as follows:

$$T = \{L_i, \ i = 1 \div 5\} = \{B, P, M, G, E\}$$

- L_1 = B (bad),
- L_2 = P (poor),
- L_3 = M (medium),
- L_4 = G (good),
- L_5 = E (excellence)

For the sake of simplicity, the model defines linguistic quality levels in the linguistic set T by triangular fuzzy numbers in the base set X as shown in **Figure 2**.

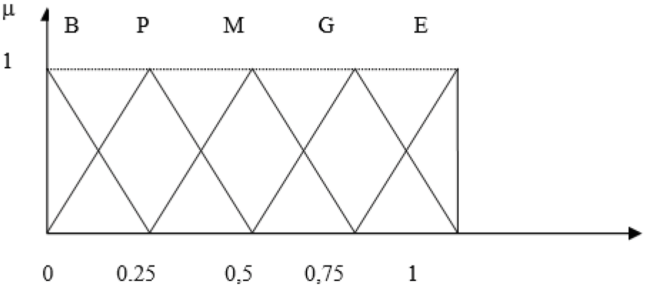

FIGURE 2 | The linguistic quality levels.

- B = (0, 0, 0.25),
- P = (0.25, 0.25, 0.25),
- M = (0.5, 0.25, 0.25),
- G = (0.75, 0.25, 0.25),
- E = (1, 0.25, 0)

3.2. Step 2: Collect data on the quality characteristic under control

In order to analyze and develop the control chart, the model collects m samples with sample size n, each sample having n observations. A data sample Sj could be described as follows:

$$S_j = \{(L_i, k_{ij} \mid \mathrm{i} = 15\}, \mathrm{j} = 1 \div m$$

where k_{ij} is the number of observations of linguistic quality level L_i in the j^{th} sample S_j.

The total number of observations in each sample:

$$k_{1j} + k_{2j} + +k_{5j} = n, \mathrm{j} = 1 \div m$$

3.3. Step 3: Construct the fuzzy variable of sample mean $\overline{X}_j$

The sample means are defined as follows.

$$\overline{X}_j = \frac{L_1 k_{1j} + L_2 k_{2j} + +L_5 k_{5j}}{n}, j = 1 \div m$$

The linguistic quality levels are defined as triangular numbers:

$$L_i = \left(a_i, b_{i,} c_i\right), i = 1 \div 5$$

According to the property of triangular numbers, the sample means are also triangular fuzzy variables.

$$\overline{X}_j = (A_j, B_j, C_j)$$

$$A_j = \frac{a_1 k_{1j} + a_2 k_{2j} + \ldots + a_t k_{5j}}{n}$$

$$B_j = \frac{b_1 k_{1j} + b_2 k_{2j} + \ldots + b_t k_{5j}}{n}$$

$$C_j = \frac{c_1 k_{1j} + c_2 k_{2j} + \ldots + c_t k_{5j}}{n}$$

3.4. Step 4: Identify the fuzzy variable of grand sample means

The grand sample mean is defined as follows.

$$\overline{\overline{X}} = \frac{\sum_{j=1}^{m} \overline{X}_j}{m}$$

Because sample means are triangular variables, $\overline{X}_j = (A_j, B_j, C_j)$, the grand sample mean is also a triangular variable:

$$\overline{\overline{X}} = (A, B, C)$$

$$A = \frac{\sum_{j=1}^{m} A_j}{m}, B = \frac{\sum_{j=1}^{m} B_j}{m}, C = \frac{\sum_{j=1}^{m} C_j}{m}$$

3.5. Step 5: Identify the center line CL

The CL is the expected value of the grand sample mean. Because the grand is a triangular variable (A, B, C), the CL is defined as follows:

$$CL = \mu_{\overline{\overline{X}}} = A$$

3.6. Step 6: Identify the grand sample mean's standard deviation $\sigma_{\overline{\overline{X}}}$

The grand mean $\overline{\overline{X}}$ is a triangular variable (A, B, C). According to Kauffman & Gupta:

$$\sigma_{\overline{\overline{X}}} = (B + C) * \frac{1}{2} = \frac{B + C}{2}$$

3.7. Step 7: Construct the control limits UCL and LCL

According to Shewhart's principle, with relative distance L, UCL and LCL are defined as follows:

$$LCL = Max\left[0, \left(\mu_{\overline{\overline{X}}} - L\sigma_{\overline{\overline{X}}}\right)\right]$$

$$UCL = Min\left[1, \left(\mu_{\overline{\overline{X}}} + L\sigma_{\overline{\overline{X}}}\right)\right]$$

Or

$$LCL = Max\left[0, \left(A - \frac{L\,(B + C)}{2}\right)\right]$$

$$UCL = Min\left[1, \left(A + \frac{L\,(B + C)}{2}\right)\right]$$

3.8. Step 8: Identify the sample points $\overline{X}_j$ in the chart

The sample points in the chart have the values of the expected values of the sample means $\overline{X}_j$. Because $\overline{X}_j$, j = 1÷m are triangular fuzzy variables $\overline{X}_j = (A_j, B_j, C_j)$, then

$$X_j = \mu_{\overline{X}_j} = A_j$$

TABLE 1 | The linguistic quality levels defined as triangular number.

i	a_i	b_i	c_i
1	0	0	0.25
2	0.25	0.25	0.25
3	0.5	0.25	0.25
4	0.75	0.25	0.25
5	1	0.25	0

3.9. Step 9: Assess the control status of the process

The control chart is drawn with control limits and sample points. If all the sample points are inside the control limits, the process is in control. If a point is outside the limits, the cause must be found. If there is an external cause, then remove this point, recalculate the control limits, until all points are within the limits, or outside the limits, without any external cause (1).

4. A numerical case

To illustrate the model, a numerical case is shown via the following steps.

4.1. Step 1: Define quality levels of the linguistic variables

The quality levels of the linguistic variable are defined by the following linguistic set T.

$$T = \{L_i,\ i = 1 \div 5\} = \{B,\ P,\ M,\ G,\ E\}$$

The linguistic quality levels L_i are defined as triangular numbers (a_i, b_i, c_i), i = 1÷5, shown in **Table 1**.

- L_1 = B = (0, 0, 0.25),
- L_2 = P = (0.25, 0.25, 0.25),
- L_3 = M = (0.5, 0.25, 0.25),
- L_4 = G = (0.75, 0.25, 0.25),
- L_5 = E = (1, 0.25, 0)

4.2. Step 2: Collect data on the quality characteristics under control

The collected data include 30 samples with a sample size of 10, as shown in **Table 2**.

The numbers of observations k_{ij}, i = 1÷5, j = 1÷30 of linguistic quality levels L_i in the sample S_j are as in **Table 3**.

TABLE 2 | The collected data with 30 samples.

S_j	B	P	M	G	E	S_j	B	P	M	G	E
S_1	2	1	7	0	0	S_{16}	0	3	6	0	1
S_2	1	2	6	1	0	S_{17}	1	5	2	2	0
S_3	0	3	6	1	0	S_{18}	3	5	1	1	0
S_4	0	8	1	1	0	S_{19}	1	5	4	0	0
S_5	0	8	2	0	0	S_{20}	0	5	3	2	0
S_6	1	5	2	2	0	S_{21}	4	4	2	0	0
S_7	0	5	5	0	0	S_{22}	0	6	2	2	0
S_8	1	6	3	0	0	S_{23}	0	4	6	0	0
S_9	0	6	4	0	0	S_{24}	1	0	8	1	0
S_{10}	1	3	3	2	1	S_{25}	0	5	5	0	0
S_{11}	0	5	4	1	0	S_{26}	0	5	5	0	0
S_{12}	1	0	7	2	0	S_{27}	0	3	6	1	0
S_{13}	0	4	6	0	0	S_{28}	0	3	6	0	1
S_{14}	0	4	6	0	0	S_{29}	0	3	7	0	0
S_{15}	0	8	1	1	0	S_{30}	1	4	5	0	0

4.3. Step 3: Construct the fuzzy variable of sample mean $\overline{X}_j$

The sample means $\overline{X}_j$ are triangular fuzzy variables (A_j, B_j, C_j)

$$A_j = \frac{a_1 k_{1j} + a_2 k_{2j} + a_3 k_{3j} + a_4 k_{4j} + a_5 k_{5j}}{10}$$

$$B_j = \frac{b_1 k_{1j} + b_2 k_{2j} + b_3 k_{3j} + b_4 k_{4j} + b_5 k_{5j}}{10}$$

$$C_j = \frac{c_1 k_{1j} + c_2 k_{2j} + c_3 k_{3j} + c_4 k_{4j} + c_5 k_{5j}}{10}$$

By substituting the values of a_i, b_i, c_i, we get the following result.

$$A_j = \frac{0.25k_{2j} + 0.5k_{3j} + 0.75k_{4j} + k_{5j}}{10} = 0.025(k_{2j} + 2k_{3j} + 3k_{4j} + 4k_{5j})$$

$$B_j = \frac{0.25k_{2j} + 0.5k_{3j} + 0.75k_{4j} + k_{5j}}{10} = 0.025(k_{2j} + 2k_{3j} + 3k_{4j} + 4k_{5j})$$

$$C_j = \frac{0.25k_{2j} + 0.5k_{3j} + 0.75k_{4j} + k_{5j}}{10} = 0.025(k_{2j} + 2k_{3j} + 3k_{4j} + 4k_{5j})$$

The values of the sample means are calculated as shown in **Table 4**.

TABLE 3 | The numbers of observations of linguistic quality levels in the samples.

j	k_{1j}	k_{2j}	k_{3j}	k_{4j}	k_{5j}	j	k_{1j}	k_{2j}	k_{3j}	k_{4j}	k_{5j}
1	2	1	7	0	0	16	0	3	6	0	1
2	1	2	6	1	0	17	1	5	2	2	0
3	0	3	6	1	0	18	3	5	1	1	0
4	0	8	1	1	0	19	1	5	4	0	0
5	0	8	2	0	0	20	0	5	3	2	0
6	1	5	2	2	0	21	4	4	2	0	0
7	0	5	5	0	0	22	0	6	2	2	0
8	1	6	3	0	0	23	0	4	6	0	0
9	0	6	4	0	0	24	1	0	8	1	0
10	1	3	3	2	1	25	0	5	5	0	0
11	0	5	4	1	0	26	0	5	5	0	0
12	1	0	7	2	0	27	0	3	6	1	0
13	0	4	6	0	0	28	0	3	6	0	1
14	0	4	6	0	0	29	0	3	7	0	0
15	0	8	1	1	0	30	1	4	5	0	0

TABLE 4 | The values of the sample means.

j	A_j	B_j	C_j	j	A_j	B_j	C_j
1	0.375	0.2	0.25	16	0.475	0.25	0.225
2	0.425	0.225	0.25	17	0.375	0.225	0.25
3	0.45	0.25	0.25	18	0.25	0.175	0.25
4	0.325	0.25	0.25	19	0.325	0.225	0.25
5	0.3	0.25	0.25	20	0.425	0.25	0.25
6	0.375	0.225	0.25	21	0.2	0.15	0.25
7	0.375	0.25	0.25	22	0.4	0.25	0.25
8	0.3	0.225	0.25	23	0.4	0.25	0.25
9	0.35	0.25	0.25	24	0.475	0.225	0.25
10	0.475	0.225	0.225	25	0.375	0.25	0.25
11	0.4	0.25	0.25	26	0.375	0.25	0.25
12	0.5	0.225	0.25	27	0.45	0.25	0.25
13	0.4	0.25	0.25	28	0.475	0.25	0.225
14	0.4	0.25	0.25	29	0.425	0.25	0.25
15	0.325	0.25	0.25	30	0.35	0.225	0.25

4.4. Step 4: Identify the fuzzy variable of grand sample means

The grand sample mean is a triangular variable:

$$\overline{\overline{X}} = (A, B, C)$$

The values of A, B, C are calculated as follows.

$$A = \frac{\sum_{j=1}^{30} A_j}{30} = 0.385$$

$$B = \frac{\sum_{j=1}^{m} B_j}{m} = 0.235$$

TABLE 5 | The values of sample points.

j	A_j	j	A_j
1	0.375	16	0.475
2	0.425	17	0.375
3	0.45	18	0.25
4	0.325	19	0.325
5	0.3	20	0.425
6	0.375	21	0.2
7	0.375	22	0.4
8	0.3	23	0.4
9	0.35	24	0.475
10	0.475	25	0.375
11	0.4	26	0.375
12	0.5	27	0.45
13	0.4	28	0.475
14	0.4	29	0.425
15	0.325	30	0.35

$$C = \frac{\sum_{j=1}^{m} C_j}{m} = 0.2475$$

4.5. Step 5: Identify the center line CL

The CL is calculated as follows.

$$CL = \mu_{\overline{\overline{X}}} = A = 0.385$$

4.6. Step 6: Identify the grand sample mean's standard deviation

The grand mean's standard deviation is calculated as follows.

$$\sigma_{\overline{\overline{X}}} = \frac{B + C}{2} = 0.24125$$

4.7. Step 7: Calculate the control limits UCL and LCL

Choose the distance factor L = 0.7. Applying the model, UCL and LCL are calculated as follows.

$$LCL = Max\left[0, \left(A - \frac{L\,(B + C)}{2}\right)\right] = 0.2161$$

$$UCL = Min\left[1, \left(A + \frac{L\,(B + C)}{2}\right)\right] = 0.5539$$

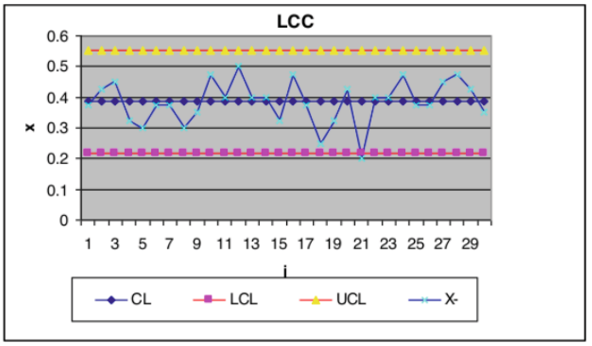

FIGURE 3 | The LCC chart.

4.8. Step 8: Identify the sample points $\overline{X}_j$ in the chart

The sample points in the chart have the values of the expected values of the sample means $\overline{X}_j$

$$X_j = \mu_{\overline{X}_j} = A_j$$

The values of sample points are calculated and shown in Table 5.

4.9. Step 9: Assess the control status of the process

The LCC chart with control limits and sample points is shown in Figure 3.

It can be seen that almost all the samples are inside the control limits, except sample 21, which is outside the control limits. Looking carefully at this sample, in 10 observations, there are 4 bad products, 4 poor products, 2 medium products, and no good or excellent products at all. The average quality level of this sample is 0,2 lower than the lower limit LCL = 0,216. The cause of this point must be found. If there is an external cause, then remove this point and recalculate the control limits, until all points are within the limits, or outside the limit, without any external cause.

5. Conclusion

The paper develops an approach to design linguistic control charts, which are more effective than traditional control charts in reducing quality costs. By modeling quality characteristics through triangular fuzzy variables, using fuzzy arithmetic calculations, as well as using Shewhart's principle for constructing control limits, the method has the advantage of simple calculation.

However, the article still has some limitations such as determining the parameter L and determining the sensitivity of the control chart. These limitations open up future research directions.

References

1. Montgomery DC. *Introduction to Statistical Quality Control*. 6th ed. Hoboken, NJ: John Wiley & Sons Inc (2009).
2. Klir GJ, Yuan B. *Fuzzy Sets and Fuzzy Logic - Theory and Applications*. Hoboken, NJ: Prentice Hall Inc (1995).
3. Nhu PN. *Fuzzy Quality Control*. Nevada: Science & Technology Press (2006).
4. Wang J, Raz T. On the construction of Control Charts using linguistic variables. *Int J Prod Res*. (1989) 28:477–87. doi: 10.1080/00207549008942731
5. Kanagawa A, Tamaki F, Ohta H. Control charts for process average and variability based on linguistic data. *Int J Prod Res*. (1993) 31:913–22. doi: 10.1080/00207549308956765
6. Franceschini F, Romano D. Control chart for linguistic variables: a method based on the use of linguistic quantifiers. *Int J Prod Res*. (1999) 37:3791–801. doi: 10.1080/002075499190059
7. Gulbay M, Kahraman C, Ruan D. α-cut fuzzy control charts for linguistic data. *Int J Intell Syst*. (2004) 19:1173–95. doi: 10.1002/int.20044
8. Dubois D, Prade H. Possibility Theory: An approach to computerized processing of uncertaity. *Int J Gen Syst*. (1989) doi: 10.1080/03081078908935040 [Epub ahead of print].