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RESEARCH

Linear solution of the three-station positioning equation reconstructed based on the path difference equation

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For a three-station path difference positioning system with an arbitrary planar layout, if the solution analysis is directly based on the path difference equation, the radial distance unknowns in the equation are difficult to directly eliminate. Once the conversion relationship between the polar coordinate system and the Cartesian coordinate system is utilized, the differential equation system can be transformed into a mixed variable equation system, which includes both the three radial distances in the polar coordinate system and the two coordinate variables in the Cartesian coordinate system. Take the radial distance of the main station as the quantity to be solved, and use the path difference equation to express the radial distance of the secondary station as a function of the radial distance of the main station. By selecting the appropriate station coordinates, an unknown variable in the Cartesian coordinate system can be eliminated. By combining mixed equations, another unknown variable in the Cartesian coordinate system can be further eliminated. Thus, a definite solution equation containing only the radial distance of the main station is obtained.

Keywords: three-station positioning, linear arrays, path difference equation, analytical equation, planar geometry, passive location

1. Introduction

The traditional multi-station passive positioning system requires solving highly nonlinear hyperbolic measurement equations. The drawback of this method is that there is no analytical solution. The positioning accuracy strongly depends on whether the initial position estimation is accurate. There will be positioning blur (1–7).

The research results of the author many years ago indicate that for the passive positioning problem of planar three stations, a linear analytical solution can be obtained using planar geometric relationships based on path difference measurement (8).

This article proves that by using the transformation relationship between Cartesian and polar coordinate systems, a mixed variable definite solution equation can be directly established based on the path difference equation, and linear analytical solutions can be obtained without the need for planar geometric relationships.

2. Existing proof

In order to facilitate the conversion of ranging solutions at different stations and enable comparison of the final derivation results, this paper has made significant changes to the identification of stations based on reference (8).

2.1. Geometric model

For a planar three-station positioning system with an arbitrary station layout, its geometric model is shown in

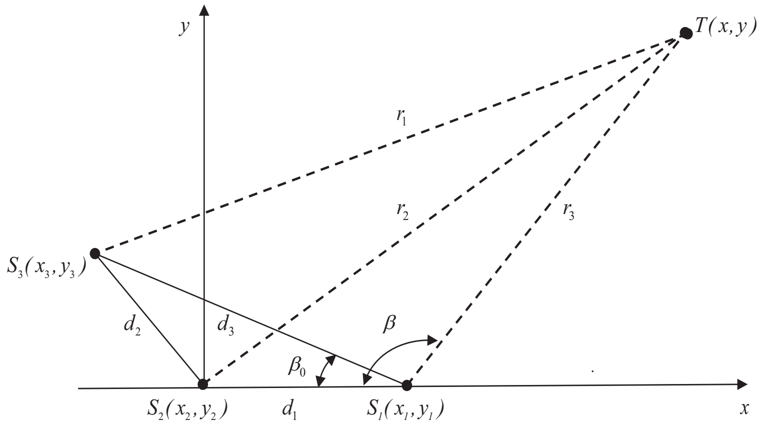

FIGURE 1 | Schematic diagram of a planar arbitrary three-station positioning array with known baseline angles.

Figure 1. Let the middle station S_2 be the main station, and the other two stations S_1 and S_3 be secondary stations. Roughly ignoring the height of distant targets, assuming that the target $T(x, y)$ is also located in a two-dimensional plane. Assumption: The angle β between the radial distance r_3 and the baseline d_1 is an unknown target azimuth angle. The baseline length d_1 and the angle β_0 between the inter-station baselines are set to values that can be measured during station deployment.

2.2 Basic solution

According to the geometric relationship shown in **Figure 1**, using the main station S_2 as the reference for path difference measurement, there is the following path difference equation:

$$\Delta r_2 = r_2 - r_3 \tag{1}$$

$$\Delta r_3 = r_3 - r_1 \tag{2}$$

where: Δr_2 is the path difference between the substation and the main station; Δr_3 the path difference between two substations.

Based on the preset angle, the cosine theorem can provide the following two trigonometric positioning equations:

$$r_2^2 = r_3^2 + d_1^2 - 2d_1 r_3 \cos\beta \tag{3}$$

$$\begin{aligned} r_1^2 &= r_3^2 + d_3^2 - 2d_3 r_3 \cos(\beta - \beta_0) \\ &= r_3^2 + d_3^2 - 2d_3 r_3 (\cos\beta\cos\beta_0 + \sin\beta\sin\beta_0) \end{aligned} \tag{4}$$

The target orientation can be solved from equation (3):

$$\cos\beta = \frac{r_3^2 + d_1^2 - r_2^2}{2d_1 r_3} \tag{5}$$

Substituting equation (5) into equation (4) yields:

$$\begin{aligned} r_1^2 - r_3^2 = d_3^2 - 2d_3 r_3 \Bigg[\left(\frac{r_3^2+d_1^2-r_2^2}{2d_1 r_3}\right)\cos\beta_0 + \\ \sqrt{1-\left(\frac{r_3^2+d_1^2-r_2^2}{2d_1 r_3}\right)^2}\sin\beta_0 \Bigg] \end{aligned} \tag{6}$$

From the relationship equation of path difference: $\Delta r_2 = r_2 - r_3$, we obtain:

$$r_3^2 - r_2^2 = -2\Delta r_2 r_3 - \Delta r_2^2 \tag{7}$$

From the relationship equation of path difference: $r_1 = r_3 - \Delta r_3$, we obtain:

$$r_1^2 = r_3^2 - 2\Delta r_3 r_3 + \Delta r_3^2 \tag{8}$$

Replace the above two equations with equation (6):

$$\begin{aligned} &-2\Delta r_3 r_3 + \Delta r_3^2 \\ &= d_3^2 - 2d_3 r_3 \Bigg[\left(\frac{d_1^2 - 2\Delta r_2 r_3 - \Delta r_2^2}{2d_1 r_3}\right)\cos\beta_0 + \\ &\sqrt{1-\left(\frac{d_1^2 - 2\Delta r_2 r_3 - \Delta r_2^2}{2d_1 r_3}\right)^2}\sin\beta_0 \Bigg] \end{aligned} \tag{9}$$

Thus, an analytical equation containing only one unknown variable r_3 is obtained.

2.3 Transformation processing

After transforming equation (9), there are:

$$\begin{aligned} &2d_3 r_3 \sin\beta_0 \sqrt{1-\left(\frac{d_1^2-2\Delta r_2 r_3-\Delta r_2^2}{2d_1 r_3}\right)^2} \\ &\quad = d_3^2 - \Delta r_3^2 - \frac{d_3}{d_1}\left(d_1^2 - \Delta r_2^2\right)\cos\beta_0 + \\ &\quad 2\left(\frac{d_3}{d_1}\Delta r_2\cos\beta_0 + \Delta r_3\right) r_3 \\ &= a + b r_3 \end{aligned} \tag{10}$$

In the formula:

$$a = d_3^2 - \Delta r_3^2 - \left(d_1^2 - \Delta r_2^2\right)\frac{d_3}{d_1}\cos\beta_0$$

$$b = 2\left(\frac{d_3}{d_1}\Delta r_2\cos\beta_0 + \Delta r_3\right)$$

After squaring both sides of equation (10), we obtain:

$$\begin{aligned}&4d_3^2r_3^2\sin^2\beta_0-\frac{d_3^2}{d_1^2}\sin^2\beta_0\left(d_1^2-2\Delta r_2r_3-\Delta r_2^2\right)^2\\&=a^2+2abr_3+b^2r_3^2\end{aligned}\tag{11}$$

If $c=d_1^2-\Delta r_2^2$, there are:

$$\begin{aligned}&4d_3^2r_3^2\sin^2\beta_0-\frac{d_3^2}{d_1^2}\sin^2\beta_0\left(c^2-4c\Delta r_2r_3+4\Delta r_2^2r_3^2\right)\\&=a^2+2abr_3+b^2r_3^2\end{aligned}\tag{12}$$

After expansion, there are:

$$\begin{aligned}&4d_3^2r_3^2\sin^2\beta_0-c^2\frac{d_3^2}{d_1^2}\sin^2\beta_0+4c\frac{d_3^2}{d_1^2}\sin^2\beta_0\Delta r_2r_3\\&-4\Delta r_2^2\frac{d_3^2}{d_1^2}\sin^2\beta_0r_3^2\\&=a^2+2abr_3+b^2r_3^2\end{aligned}\tag{13}$$

Finally, a quadratic equation of one variable is obtained:

$$\begin{aligned}&a^2+c^2\frac{d_3^2}{d_1^2}\sin^2\beta_0+(2ab-4c\Delta r_2\frac{d_3^2}{d_1^2}\sin^2\beta_0)r_3\\&+(b^2+4\Delta r_2^2\frac{d_3^2}{d_1^2}\sin^2\beta_0-4d_3^2\sin^2\beta_0)r_3^2=0\end{aligned}\tag{14}$$

2.4 Degradation verification

Equation (14) can be labeled as:

$$Ar_1^2+Br_1+C=0\tag{15}$$

In the formula:

$$\begin{aligned}A&=b^2+4\Delta r_2^2\frac{d_3^2}{d_1^2}\sin^2\beta_0-4d_3^2\sin^2\beta_0\\&=4\left(\frac{d_3}{d_1}\Delta r_2\cos\beta_0+\Delta r_3\right)^2+4\Delta r_2^2\frac{d_3^2}{d_1^2}\sin^2\beta_0-4d_3^2\sin^2\beta_0\end{aligned}$$

$$\begin{aligned}B&=2ab-4c\Delta r_1\frac{d_3^2}{d_1^2}\sin^2\beta_0\\&=4\left(d_3^2-\Delta r_3^2-\left(d_1^2-\Delta r_2^2\right)\frac{d_3}{d_1}\cos\beta_0\right)\\&\left(\frac{d_3}{d_1}\Delta r_2\cos\beta_0+\Delta r_3\right)-4\left(d_1^2-\Delta r_2^2\right)\Delta r_1\frac{d_3^2}{d_1^2}\sin^2\beta_0\end{aligned}$$

$$\begin{aligned}C=a^2+c^2\frac{d_3^2}{d_1^2}\sin^2\beta_0&=\left(d_3^2-\Delta r_3^2-\left(d_1^2-\Delta r_2^2\right)\frac{d_3}{d_1}\cos\beta_0\right)^2\\&+\left(d_1^2-\Delta r_2^2\right)^2\frac{d_3^2}{d_1^2}\sin^2\beta_0\end{aligned}$$

Once the angle β_0 approaches zero, that is, when the three stations are arranged in a straight line, equation (14) will degenerate into:

$$a^2+2abr_3+b^2r_3^2=0\tag{16}$$

That is: $a+br_3=0$, for a symmetric equal-length array, i.e. when $d_3=2d_1=2d$, we can obtain:

$$r_3=-\frac{a}{b}=-\frac{4d^2-\Delta r_3^2-2\left(d^2-\Delta r_2^2\right)}{2\left(2\Delta r_2+\Delta r_3\right)}\tag{17}$$

2.5 Ranging solution for the midpoint of the array

In order to facilitate comparison with the derivation results in the next chapter, this section converts the ranging solution at the third station to the ranging solution at the midpoint of the array.

Order:

$$\Delta r_1=r_1-r_2\tag{18}$$

Because: $\Delta r_3=r_3-r_1$, there is: $\Delta r_3=r_3-r_2+r_2-r_1=-\Delta r_2-\Delta r_1$. Because of: $\Delta r_2=r_2-r_3$, there is: $r_3=r_2-\Delta r_2$. Substitute these equations into equation (17):

$$r_2-\Delta r_2=-\frac{4d^2-(\Delta r_2+\Delta r_1)^2-2\left(d^2-\Delta r_2^2\right)}{2\left(2\Delta r_2-\Delta r_2-\Delta r_1\right)}\tag{19}$$

We can obtain:

$$r_2=\frac{2d^2-\Delta r_1^2-\Delta r_2^2}{2\left(\Delta r_1-\Delta r_2\right)}\tag{20}$$

This result is completely consistent with the analytical result obtained when arranged in a straight line with three stations (8). This verification result indicates that for a three-station positioning system with an arbitrary layout in a plane, analytical results can be obtained without complex nonlinear operations using additional geometric conditions.

3. Mixed analysis method of double coordinate systems

3.1 Geometric structure and distance equation

First, assume there are any three stations on the two-dimensional plane, and the geometric figure is shown in **Figure 2**. Preset the station S_2 as the main station, and the other two stations S_1 and S_3 as the secondary stations.

In the polar coordinate system, two path difference equations (18) and (1) can be obtained from the three stations. In the Cartesian coordinate system, the radial distance can be expressed as:

$$r_1=\sqrt{(x-a_1)^2+(y-b_1)^2}\tag{21}$$

$$r_2=\sqrt{(x-a_2)^2+(y-b_2)^2}\tag{22}$$

$$r_3=\sqrt{(x-a_3)^2+(y-b_3)^2}\tag{23}$$

In the formula, a_i and b_i are the coordinates of each station, respectively.

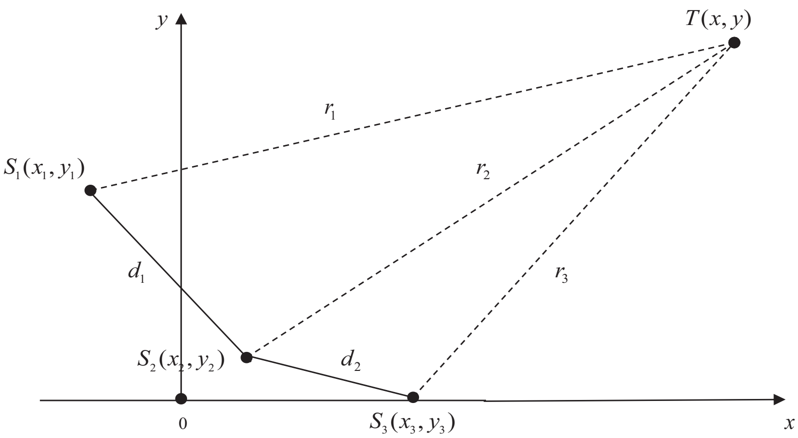

FIGURE 2 | Geometric schematic diagram of a planar arbitrary three-station positioning array based on mixed equations.

3.2 Underdetermined solution

First, square the path difference equations after moving the terms, expand them, and then move the terms again to have:

$$\Delta r_{12}^2+2\Delta r_{12}r_2 = r_1^2 - r_2^2 \tag{24}$$

$$\Delta r_{23}^2 + 2\Delta r_{23}r_3 = r_2^2 - r_3^2 \tag{25}$$

Expand the radial distances on the right side of each equation with the expression in the Cartesian coordinate system:

$$\begin{aligned}\Delta r_{12}^2+2r_2\Delta r_{12} &= (x-a_1)^2 + \left(y-b_1\right)^2 - (x-a_2)^2 \\ &- \left(y-b_2\right)^2 \\ &= 2x\left(a_2-a_1\right) + 2y\left(b_2-b_1\right) + a_1^2 - a_2^2 + b_1^2 - b_2^2\end{aligned} \tag{26}$$

$$\begin{aligned}\Delta r_{23}^2+2r_3\Delta r_{23} &= (x-a_2)^2 + \left(y-b_2\right)^2 - (x-a_3)^2 \\ &- \left(y-b_3\right)^2 \\ &= 2x\left(a_3-a_2\right) + 2y\left(b_3-b_2\right) + a_2^2 - a_3^2 + b_2^2 - b_3^2\end{aligned} \tag{27}$$

Further use the path difference equation to transform the radial distance r_3 on the left side of equation (27) into a parametric function related to the main station's radial distance:

$$\begin{aligned}\Delta r_{23}^2+2\left(r_2-\Delta r_{23}\right)\Delta r_{23} &= 2\Delta r_{23}r_2 - \Delta r_{23}^2 \\ = 2x\left(a_3-a_2\right) + 2y\left(b_3-b_2\right) &+ a_2^2 - a_3^2 + b_2^2 - b_3^2\end{aligned} \tag{28}$$

In the case of any three sites, only two mixed definite solution equations can be obtained, while there are three unknowns to be solved.

3.3 Definite solution equation

If the coordinate parameters in the y-axis are all the same, the variable y will be eliminated. At this time, the array is a linear array along the x-axis. For simplicity, directly make b_i equal to zero, then there is:

$$2\Delta r_{12}r_2+\Delta r_{12}^2 = 2x\left(a_2-a_1\right) + a_1^2 - a_2^2 \tag{29}$$

$$2\Delta r_{23}r_2 - \Delta r_{23}^2 = 2x\left(a_3-a_2\right) + a_2^2 - a_3^2 \tag{30}$$

For clarity, first transform the above equations. Set equation (29) as the first definite solution equation:

$$A_1x = F_1 \tag{31}$$

In the formulas:

$$A_1 = 2\left(a_2-a_1\right)$$

$$F_1 = 2\Delta r_{12}r_2+\Delta r_{12}^2 - \left(a_1^2-a_2^2\right)$$

Set equation (30) as the second definite solution equation:

$$A_2x{=}F_2 \tag{32}$$

In the formulas:

$$A_2 = 2\left(a_3-a_2\right)$$

$$F_2 = 2\Delta r_{23}r_2 - \Delta r_{23}^2 - \left(a_2^2-a_3^2\right)$$

Multiply A_2 on both sides of the first equation first: $A_1A_2x = A_2F_1$. Multiplying A_1 on both sides of the second equation: $A_1A_2x{=}A_1F_2$. After subtraction, there is:

$$A_2F_1 - A_1F_2 = 0 \tag{33}$$

After expansion and organization, it can be seen that the existing mixed definite solution equations have been transformed into a definite solution equation that only contains the main station's radial distance:

$$\begin{aligned}&2\left(A_2\Delta r_{12} - A_1\Delta r_{23}\right)r_2 = \\ &A_2\left(a_1^2-a_2^2\right) - A_1\left(a_2^2-a_3^2\right) - A_2\Delta r_{12}^2 - A_1\Delta r_{23}^2\end{aligned} \tag{34}$$

From which the radial distance r_2 can be obtained:

$$r_2 = \frac{A_2\left(a_1^2-a_2^2\right) - A_1\left(a_2^2-a_3^2\right) - A_2\Delta r_{12}^2 - A_1\Delta r_{23}^2}{\left(2A_2\Delta r_{12} - 2A_1\Delta r_{23}\right)} \tag{35}$$

3.4 Qualitative analysis

If the main station is set at the origin of the Cartesian coordinate system and an equidistant linear array is considered, assuming the length of the array baseline is d, that is:

$$a_2{=}0, a_1 = -d, a_3 = d.$$

Once these results are put into formula (35), there is:

$$r_2 = \frac{2d^2 - \Delta r_{12}^2 - \Delta r_{23}^2}{2\left(\Delta r_{12} - \Delta r_{23}\right)} \tag{36}$$

This is the existing linear solution of the one-dimensional equidistant double-base linear array.

4. Conclusion

For the problem of plane three-station positioning, analytical solutions can be obtained by utilizing existing plane geometric relationships or by utilizing the transformation relationship between two coordinate systems. For multi-station positioning systems, the positioning error of the target is closely related to its relative position to each measurement station (9, 10). Optimizing the layout of measurement stations under certain error factors is an effective means to improve positioning accuracy. The analysis in this article will contribute to further in-depth research and optimization of the station configuration problem in multi-station positioning systems.

From a step-by-step perspective, the deduction many years ago was actually based on geometric equations as the main approach. The path differential equation is auxiliary. It only used the path difference equation to eliminate the unknowns in the geometric equations. Therefore, the original proof should more rigorously be called a solution method based on geometric equations.

This paper proves the path difference equation as the main, where the unknown radial distance is difficult to eliminate directly. However, once the transformation relationship between the polar coordinate system and the Cartesian coordinate system is used, the path difference equation can be transformed into a mixed variable equation that is more suitable for offsetting unknown variables. As a result, the linear solution of the non-equidistant double-base array can be directly obtained without using planar geometry.

Similar to the solution method of the path difference equation based on the transformation of two-coordinate systems, the solution method using geometric equations can also provide linear solutions for the positioning of any three stations on a plane, but the mathematical derivation process is slightly more complicated. As far as the final expression form— the linear solution of a linear array—is concerned, there seems to be no difference between the two deduction methods. However, the solution results of geometric equations directly include directional parameters, while the solution method of the path difference equation based on two-coordinate system transformation is only related to coordinate values. Therefore, the author personally believes that once the number of detection stations on the plane is increased, from the perspectives of simplifying the complexity of the problem, facilitating calculation, and supporting software tools, the solution method based on the path difference equation of the two-coordinate system transformation, which is only related to the coordinate values of the Cartesian coordinate system, will be more conducive to using the matrix method for processing.

References

1. Caffery JJ, Stuber GL. Overview of radio location in CDMA cellular systems. *IEEE Commun Mag.* (1998) 36:38–45.
2. Wang Y, Zhang L. Research on multi-station time difference location technology. *Modern Radar.* (2003) 25:1–4. doi: 10.3969/j.issn.1004-7859.2003.02.001
3. Li W, Zhang Z, Li H, et al. Research on performance test evaluation technology of multi-station time difference location. *Syst Eng Theory and Practice.* (2015) 35:7.
4. Zhang Z. *Research on Passive Location of Radiation Sources (Ph.D. dissertation.* Xidian: Xidian University (2000).
5. Liu G. *Research on Distributed Multi-station Passive Time Difference Location System Ph.D. dissertation.* Xidian: Xidian University (2006).
6. Ren L. *Research on Time Measurement and Passive Multi-station Location Method Ph.D. dissertation.* Harbin: Harbin Institute of Technology (2006).
7. Wang C, Li S, Huang H. Location accuracy analysis and optimal distribution of TOA difference location system. *Fire Control Radar Technol.* (2003) 32:1–6.
8. Yu T. *Passive Detection and Location Technology.* Beijing: National Defense Industry Press (2017).
9. Qian Y, Lu M, Feng Z. Research on HDOP in ground positioning system based on TDOA Principle. *Telecommun Technol.* (2005) 45:135–8.
10. Wang C, Li S, Huang H. Analysis of positioning accuracy and optimal station layout in time-difference-of-arrival positioning system. *Fire Control Radar Technol.* (2003) 32:1–6.